arXiv:funct-an/9412004v2 11 Jan 1995

Diagonalization of compact operators in Hilbert modules over finite W^*-algebras

V. M. Manuilov

22 July 1994

Abstract

It is known that a continuous family of compact operators can be diagonalized pointwise. One can consider this fact as a possibility of diagonalization of the compact operators in Hilbert modules over a commutative W^*-algebra. The aim of the present paper is to generalize this fact for a finite W^*-algebra A not necessarily commutative. We prove that for a compact operator K acting in the right Hilbert A-module H_A^* dual to H_A under slight restrictions one can find a set of "eigenvectors" $x_i \in H_A^*$ and a non-increasing sequence of "eigenvalues" $\lambda_i \in A$ such that $K\, x_i = x_i\, \lambda_i$ and the autodual Hilbert A-module generated by these "eigenvectors" is the whole H_A^*. As an application we consider the Schrödinger operator in magnetic field with irrational magnetic flow as an operator acting in a Hilbert module over the irrational rotation algebra A_θ and discuss the possibility of its diagonalization.

1 Introduction

The classical Hilbert-Schmidt theorem states that any compact self-adjoint operator acting in a Hilbert space can be diagonalized. It is also known that a continuous family of compact operators is diagonalizable. When active study of Hilbert modules began some results were obtained concerning diagonalizability of some operators acting in these modules. R. V. Kadison [8],[9] proved that a self-adjoint operator in a free finitely generated module over a W^*-algebra is diagonalizable. Later on some other interesting results about diagonalization of operators appeared [7],[16],[24]. This paper is a step in the same direction and is concerned with diagonalization of compact operators in the Hilbert module H_A^* over a finite W^*-algebra A. Its main results were announced in [15].

The present paper is organized as follows: At section 2 we study some properties of Hilbert modules over finite W^*-algebras related with orthogonal complementability. The main technical result is the isomorphy of H_A^* and the

orthogonal complement to A in H_A^*. At section 3 we recall the basic facts about the compact operators in Hilbert modules. Here we also give an example showing that the module H_A is not sufficient to diagonalize compact operators, so we must turn to its dual module H_A^*. Section 4 contains the proof of the main theorem of this paper about diagonalization of a compact operator in the module H_A^*. Here we also discuss the uniqueness condition for the "eigenvalues" of this operator. Section 5 deals with quadratic forms on Hilbert modules related to a self-adjoint operator. Properties of these forms are mostly the same as on a Hilbert space. At section 6 we discuss an example which motivated the present paper. We consider the perturbated Schrödinger operator with irrational magnetic flow as an operator acting in a Hilbert module over the irrational rotation algebra A_θ and we show that this operator is diagonalizable.

Acknoledgement. This work was partially supported by the Russian Foundation for Fundamental Research (grant N 94-01-00108-a) and the International Science Foundation (grant N MGM000). I am indebted to M. Frank, A. A. Irmatov, A. S. Mishchenko and E. V. Troitsky for helpful discussions.

2 Orthogonal complements in Hilbert modules over finite W^*-algebras

Throughout this paper A is a finite W^*-algebra admitting the central decomposition into a direct integral over a compact Borel space. By τ we denote a normal faithful finite trace on A with $\tau(1) = 1$. Recall some facts about Hilbert modules. Standart references on them are [10],[12],[19]. If B is a C^*-algebra we denote by H_B (another usual denotation is $l_2(B)$) the right Hilbert B-module consisting of the sequences (x_i), $i \in \mathbf{N}$ for which the series $\sum_i x_i^* x_i$ converges in the norm topology in B with the inner product $\langle x, y\rangle = \sum_i x_i^* y_i$ and the norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$. Let H_B^* be its dual module, $H_B^* = \mathrm{Hom}_B(H_B; B)$. It is shown in [20] that in the case of W^*-algebras the inner product on the module H_B can be prolonged to the inner product on the module H_B^* and this module is autodual, i.e. $(H_B^*)^* = H_B^*$.

Let $M \subset H_B^*$ be a Hilbert B-submodule. By $M^\perp$ we denote its orthogonal complement in H_B^*. It is well-known [3] that if M is a finitely generated projective Hilbert B-submodule in H_B^* then it is orthogonally complementary: $H_B^* = M \oplus M^\perp$. If we change H_B^* by H_B then the orthogonal complement to M in H_B is isomorphic to H_B, but nothing is known in general about isomorphy between $M^\perp$ and H_B^*. The following theorem solves this problem in the case of modules over a W^*-algebra decomposable into a direct integral of finite factors and having a faithful finite trace.

Theorem 2.1. *If M is a finitely generated projective A-submodule in H_A^* then $M^\perp$ is isomorphic to H_A^*.*

Proof. The idea of the following proof is contained in [3]. Let $g_1, \dots, g_n$ be generators of the module M. Without loss of generality we can assume that the operators $\langle g_i, g_i \rangle \in A$ are projections, $\langle g_i, g_i \rangle = p_i$. Let $\{e_m\}$ be the standart basis of the module $H_A \subset H_A^*$. Fix $\varepsilon < 0$ and define elements $e'_m \in M^\perp$ by the equality

$$e'_m = e_m - \sum_{i=1}^{n} g_i \langle g_i, e_m \rangle.$$

Then we have

$$\langle e'_m, e'_m \rangle = 1 - \sum_{i=1}^{n} \langle g_i, e_m \rangle^* \langle g_i, e_m \rangle.$$

It follows from the equality

$$\tau\Big(\langle g_i, g_i \rangle\Big) = \tau\Big(\sum_m \langle g_i, e_m \rangle^* \langle g_i, e_m \rangle\Big)$$

that the series $\sum_m \tau\Big(\langle g_i, e_m \rangle^* \langle g_i, e_m \rangle\Big)$ converges and there exists such number m_0 that for any $m > m_0$ the inequalities

$$\tau\Big(\langle g_i, e_m \rangle^* \langle g_i, e_m \rangle\Big) < \frac{\varepsilon}{2n};$$

$$\tau\Big(\langle e'_m, e'_m \rangle\Big) > 1 - \frac{\varepsilon}{2}$$

hold.

Lemma 2.2. *If $x \in H_A^*$, $\|x\| = 1$ and $\tau\left(\langle x, x \rangle\right) > 1 - \frac{\varepsilon}{2}$ then there exists a projection $p \in A$ with $\tau(p) > 1 - \varepsilon$ such that $p \langle x, x \rangle p$ is an invertible operator in the W^*-algebra pAp.*

Proof. Let $dP(\lambda)$ denote the projection-valued measure for the operator $a = \langle x, x \rangle \in A$; $a = \int_0^1 \lambda dP(\lambda)$. Put

$$f(\lambda) = \begin{cases} 0, & \lambda \le \lambda_0, \\ 1, & \lambda > \lambda_0, \end{cases}$$

where $\lambda_0 \in [0; 1]$. Then $f(a) = p$ is a projection. Denote $d\tau\left(P(\lambda)\right)$ by $d\mu(\lambda)$. It is a usual measure on $[0; 1]$ and by [17]

$$\tau(a) = \int_0^1 \lambda \, d\mu(\lambda).$$

We have

$$1 - \frac{\varepsilon}{2} < \tau(a) = \int_0^1 \lambda \, d\mu(\lambda) \le \lambda_0 \, \mu\left([0; \lambda_0)\right) + \mu\left([\lambda_0; 1]\right). \tag{2.1}$$

Since $P(1) = 1$ we have

$$\mu\left([0;\lambda_0)\right) + \mu\left([\lambda_0;1]\right) = 1 \tag{2.2}$$

From (2.1) and (2.2) we obtain the inequality

$$\mu\left([\lambda_0;1]\right) > 1 - \frac{\varepsilon}{2\left(1-\lambda_0\right)}.$$

Choosing an appropriate number $\lambda_0 \neq 0$ we obtain $\mu\left([\lambda_0;1]\right) > 1-\varepsilon$. From the definition of the function $f(\lambda)$ we have

$$\begin{aligned} \tau(p) &= \tau\left(f(\lambda)\right) &= \int_0^1 f(\lambda)\, d\mu(\lambda) \\ &= \int_{\lambda_0}^1 d\mu(\lambda) &= \mu\left([\lambda_0;1]\right) > 1-\varepsilon. \end{aligned}$$

Consider now the operator $pap \in pAp$. The equality

$$pap = \int_0^1 \lambda\, f(\lambda)\, dP(\lambda)$$

follows from the spectral theorem, therefore the spectrum of the operator pap as an element of the W^*-algebra pAp lies in $[\lambda_0;1]$, hence is separated from zero and this operator is invertible in pAp. $\bullet$

Let

$$\Big(p\langle e'_m, e'_m\rangle p\Big)^{-1/2} = pbp = b \in pAp$$

and $e''_m = e'_m \cdot b$. Then

$$\langle e''_m, e''_m\rangle = pbp\langle e'_m, e'_m\rangle pbp = p.$$

Now take an element $y \in M^\perp$, $y \neq 0$, $\|y\| \leq 1$. For every $\varepsilon > 0$ beginning from a certain number m we have

$$\begin{aligned} \tau\Big(\langle e''_m, y\rangle^* \langle e''_m, y\rangle\Big) &= \tau\left(\langle e_m, y\rangle^* b^2 \langle e_m, y\rangle\right) \\ &\leq \|b\|^2 \tau\Big(\langle e_m, y\rangle^* \langle e_m, y\rangle\Big) < \frac{\varepsilon^2}{2} \end{aligned}$$

because of convergence of the series $\sum_m \tau\Big(\langle e_m, y\rangle^* \langle e_m, y\rangle\Big)$.

Denote the operator $\langle e''_m, y\rangle^* \langle e''_m, y\rangle \in A$ by c, then $\tau(c) < \frac{\varepsilon^2}{2}$; $\|c\| \leq 1$ and $c \geq 0$ (i.e. c is a positive operator). If $dQ(\lambda)$ is its projection-valued measure and if we denote the measure $d\tau(Q(\lambda))$ by $d\nu(\lambda)$ then we have $\int_0^1 \lambda\, d\nu(\lambda) < \frac{\varepsilon^2}{2}$. If $\lambda_1 \in [0;1]$ then

$$\lambda_1 \cdot \int_{\lambda_1}^1 d\nu(\lambda) \leq \int_0^1 \lambda\, d\nu(\lambda) = \tau(c) < \frac{\varepsilon^2}{2},$$

hence $\int_{\lambda_1}^1 d\nu(\lambda) < \frac{\varepsilon^3}{2}$. Taking $\lambda_1 = \frac{\varepsilon^2}{2}$ we obtain $\nu([\lambda_1;1]) < \varepsilon$. Put

$$g(\lambda) = \begin{cases} 1, & \lambda < \lambda_1, \\ 0, & \lambda \geq \lambda_1. \end{cases}$$

Then $q = g(c)$ is a projection with $\tau(q) = \nu([0;\lambda_1)) > 1 - \varepsilon$ and

$$\|qcq\| \leq \lambda_1 = \frac{\varepsilon^2}{2}. \tag{2.3}$$

By $p \vee q$ (resp. $p \wedge q$) we denote the least upper (resp. greatest lower) bound for projections p and q. Put $p' = p \wedge q$. As by [23]

$$\tau(p) + \tau(q) = \tau(p \vee q) + \tau(p \wedge q),$$

so we have

$$\tau(p') = \tau(p) + \tau(q) - \tau(p \vee q) > (1-\varepsilon) + (1-\varepsilon) - 1 = 1 - 2\varepsilon$$

because $\tau(p \vee q) \leq \tau(1) = 1$. The inequality

$$\|p'cp'\| \leq \frac{\varepsilon^2}{2} \tag{2.4}$$

follows from (2.3). Put now $e'''_m = e''_m \cdot p'$. Then $\langle e'''_m, e'''_m \rangle = p'$. Put further $y' = y + \varepsilon e'''_m \in M^\perp$. We can decompose y' into two orthogonal summands: $y' = u + v$, where $u = y - e'''_m \langle e'''_m, y\rangle$,$v = e'''_m\Big(\langle e'''_m, y\rangle + \varepsilon \cdot 1\Big)$; $u, v \in M^\perp$. Then

$$\langle y', y'\rangle = \langle u, u\rangle + \Big(\langle e'''_m, y\rangle + \varepsilon p'\Big)^* \Big(\langle e'''_m, y\rangle + \varepsilon p'\Big)$$

and

$$\begin{aligned} p'\langle y', y'\rangle p' &= p'\langle u, u\rangle p' + \Big(p'\langle e'''_m, y\rangle p' + \varepsilon p'\Big)^* \Big(p'\langle e'''_m, y\rangle p' + \varepsilon p'\Big) \\ &= p'\langle u, u\rangle p' + \Big(\langle e'''_m, y\rangle p' + \varepsilon p'\Big)^* \Big(\langle e'''_m, y\rangle p' + \varepsilon p'\Big). \end{aligned} \tag{2.5}$$

Since

$$\begin{aligned} \Big(\langle e'''_m, y\rangle p'\Big)^* \langle e'''_m, y\rangle p' &= p'\langle e''_m, y\rangle^* p' \langle e''_m, y\rangle p' \\ &\leq p'\langle e''_m, y\rangle^* \langle e''_m, y\rangle p' \quad = \quad p'cp' \end{aligned}$$

it follows from (2.4) that $\|\langle e''_m, y\rangle p'\| \leq \frac{\varepsilon}{\sqrt{2}} < \varepsilon$. Therefore the operator $\langle e''_m, y\rangle p' + \varepsilon p'$ is invertible in the W^*-algebra $p'Ap'$. The invertibility of $p'\langle y', y'\rangle p'$ follows now from (2.5). Consider the trace norm on $M^\perp$ (and on H_A^*) defined by

$$\|x\|_\tau = \tau\Big(\langle x, x\rangle\Big)^{1/2}.$$

The inequality

$$\tau\Big(\langle y'-y,y'-y\rangle\Big)=\tau\Big(\varepsilon^2\langle e'''_m,e'''_m\rangle\Big)=\tau(\varepsilon^2p')<\varepsilon^2$$

gives us the estimate $\|y'-y\|_\tau<\varepsilon$. So we have proved that the elements of $M^\perp$ for which there exists a projection p' with $\tau(p')>\frac{1}{2}$ such that $p'\langle x,x\rangle p'$ is invertible in $p'Ap'$ are dense in $M^\perp$ in the trace norm.

Corollary 2.3. *There exists some $x\in M^\perp$ such that $\|x\|_\tau>\frac{1}{2}$ and $\|x\|\leq 1$.*

Let now $\{y_n\}$ be a sequence containing every e_m infinitely many times. Put $y=y_1-\sum_{k=1}^n g_k\langle g_k,y_1\rangle$. Then for $\varepsilon_1=1$ there exists some $y'\in M^\perp$ with $\|y'\|\leq 1$ such that

$$\|y-y'\|_\tau<\varepsilon_1 \tag{2.6}$$

and a projection p_1 with $\tau(p_1)>\frac{1}{2}$ such that $p_1\langle y',y'\rangle p_1$ is invertible in p_1Ap_1. Then putting $h_1=y'b'$ where $b'=\Big(p_1\langle y',y'\rangle p_1\Big)^{-1/2}\in p_1Ap_1$ we obtain from (2.6) the inequality

$$\mathrm{dist}_\tau(y,B_1(h_1A))\leq \mathrm{dist}_\tau(y,h_1(b')^{-1})=\mathrm{dist}_\tau(y,y')<\varepsilon_1,$$

where by B_1 we denote the unit ball of a Hilbert module in the initial norm. Therefore

$$\mathrm{dist}_\tau(y_1,B_1(\mathrm{Span}_A(M,h_1)))<\varepsilon_1.$$

Then taking $\varepsilon_2=\frac{1}{2}$ we can find an element $h_2\in(\mathrm{Span}_A(M,h_1))^\perp$ such that $\langle h_2,h_2\rangle=p_2$ is a projection with $\tau(p_2)>\frac{1}{2}$ and

$$\mathrm{dist}_\tau(y_2,B_1(\mathrm{Span}_A(M,h_1,h_2)))<\varepsilon_2.$$

Continuing this process and taking $\varepsilon_k=\frac{1}{k}$ we obtain a set of mutually orthogonal elements $h_i\in M^\perp$ with $\langle h_i,h_i\rangle=p_i$ being a projection and $\tau(p_i)>\frac{1}{2}$ such that

$$\mathrm{dist}_\tau(y_k,B_1(\mathrm{Span}_A(M,h_1,h_2,\ldots,h_k)))<\varepsilon_k \tag{2.7}$$

These h_i generate an A-module $N\subset M^\perp$ and from (2.7) we have

$$\mathrm{dist}_\tau(y_k,B_1(M\oplus N))<\frac{1}{k},$$

hence the trace norm closure of $B_1(M\oplus N)$ contains the unit ball of the whole H_A^* and the trace norm closure of $B_1(N)$ contains $B_1(M^\perp)$.

The constructed above basis $\{h_i\}$ of N is inconvenient because the inner squares of h_i are not unities. So we have to alter it. By T we denote the standart center-valued trace on A.

Lemma 2.4. *For any number C there exists some number n such that $T\left(\sum_{i=1}^{n} p_i\right) \geq C$.*

Proof. Suppose that there exists a normal state f on the center Z of A such that for some C $(f \circ T)\left(\sum_{i=1}^{\infty} p_i\right) < C$. Then there exists a central projection $z \in Z$ such that

$$T\Big(\sum_{i=1}^{\infty} p_i z\Big) < C. \tag{2.8}$$

Consider the W^*-algebra zAz. Multiplication by z turns any Hilbert module over A into a Hilbert module over zAz and preserves orthogonality of submodules. So we have $Nz \subset M^{\perp}z$ and $B_1(Nz)$ is dense in $B_1(M^{\perp}z)$ in the trace norm $\|\cdot\|_{\tau_z}$ defined by the faithful trace τ_z on zAz induced by τ. The inequality (2.8) means that for any $\varepsilon > 0$ changing z by a lesser central projection if nessessary we can find such number k that the inequality $T\left(\sum_{i>k} p_i z\right) < \varepsilon$ holds. Decompose the module Nz : $Nz = L_k \oplus R_k$ where L_k is the zAz-module generated by $h_1 z, \ldots, h_k z$ and R_k is the orthogonal complement to L_k in Nz. As $B_1(Nz)$ is dense in $B_1(M^{\perp}z)$ so $B_1(R_k)$ must be dense in $B_1((Mz \oplus L_k)^{\perp})$ in the trace norm $\|\cdot\|_{\tau_z}$. Let $x = \sum_{i>k} h_i x_i \in B_1(R_k)$. Estimate its trace norm:

$$\begin{aligned}
\|x\|_{\tau_z}^2 &= \tau_z\Big(\sum_{i>k} x_i^* \langle h_i z, h_i z\rangle x_i\Big) &&= \tau_z\Big(\sum_{i>k} x_i^* p_i z x_i\Big) \\
&= \tau_z\Big(\sum_{i>k} p_i z x_i x_i^*\Big) &&\leq \tau_z\Big(\sum_{i>k} p_i z \cdot \|x_i\|^2\Big) \\
&\leq \tau_z\Big(\sum_{i>k} p_i z \cdot \|x\|^2\Big) &&\leq \tau_z\Big(\sum_{i>k} p_i z\Big).
\end{aligned}$$

As we have $T\left(\sum_{i>k} p_i z\right) < \varepsilon$ so $\tau_z\left(\sum_{i>k} p_i z\right) < \varepsilon$ and so we obtain $\|x\|_{\tau_z}^2 < \varepsilon$ for all $x \in B_1(R_k)$. But as $B_1(R_k)$ is dense in $B_1((Mz \oplus L_k)^{\perp})$ so for all $y \in B_1((Mz \oplus L_k)^{\perp})$ we have $\|y\|_{\tau_z}^2 < \varepsilon$. On the other hand if we apply the corollary 2.3 to the module $(Mz \oplus L_k)^{\perp}$ instead of $M^{\perp}$ we can find in $B_1((Mz \oplus L_k)^{\perp})$ an element y with $\|y\|_{\tau_z} > \frac{1}{2}$. The obtained contradiction finishes the proof. •

Choose now a projection q in A with the properties:

$$T(q) = \min(T(p_1 + p_2); 1) - T(p_1) \tag{2.9}$$

and $q \perp p_1$. It follows from (2.9) that $T(q) \leq T(p_2)$, therefore there exists another projection q' equivalent to q such that $q' \leq p_2$. Equivalence of q and q' involves existance of a unitary $u \in A$ such that $qu = uq'$. Put $r = h_2 q' u^* \in N$. Then r is orthogonal to h_1 and

$$\langle r, r\rangle = uq'\langle h_2, h_2\rangle q' u^* = uq' p_2 q' u^* = uq' u^* = q.$$

Put further $H_1^{(1)} = h_1 + r$. Then

$$\langle h_1^{(1)}, h_1^{(1)} \rangle = \langle h_1, h_1 \rangle + \langle r, r \rangle = p_1 + q.$$

Notice that $T(\langle h_1^{(1)}, h_1^{(1)} \rangle) = \min(T(p_1 + p_2); 1)$. Taking into consideration the next element h_3 we can obtain $h_1^{(2)}$ such that $\langle h_1^{(2)}, h_1^{(2)} \rangle$ is a projection and $T(\langle h_1^{(2)}, h_1^{(2)} \rangle) = \min(T(p_1 + p_2 + p_3); 1)$. Repeating this procedure and increasing the value of $T(\langle h_1^{(n)}, h_1^{(n)} \rangle)$ we can construct by the lemma 2.4 an element h_1^∞ such that $\langle h_1^\infty, h_1^\infty \rangle = 1$. The orthogonal complement to h_1^∞ in N is generated by elements $h_i q_i$, $i > 1$ where q_i are some projections. Applying the construction described above to these generators we can construct by induction a set of elements h_i^∞ with $\langle h_i^\infty, h_i^\infty \rangle = 1$ which generates the module N. Hence $\{h_i^\infty\}$ is a basis in N and N is isomorphic to H_A.

Finally we must prove that $N^* = M^\perp$. As $N \subset M^\perp$ is closed in the usual norm, so for any $f \in (M^\perp)^*$ its restriction $f|_N$ belongs to N^*. Notice that the module $M^\perp$ is autodual, $(M^\perp)^* = M^\perp$ because of autoduality of H_A^* and M. Suppose that $f|_N = 0$. Since N is dense in $M^\perp$ in the trace norm, we have $f = 0$ on $M^\perp$ because of continuity of the map $f : M^\perp \longrightarrow A$ in this norm due to the inequality

$$\tau\Big((f(y))^* f(y)\Big) \leq \|f\|^2 \cdot \tau\Big(\langle y, y \rangle\Big) \tag{2.10}$$

where $y \in M^\perp$. So monomorphity of the map $M^\perp \longrightarrow N^*$ is proved. Let now $\phi \in N^*$. This functional can be prolonged to a map from $M^\perp$ to A. If $\{y_n\} \subset N$ is a sequence converging to $y \in M^\perp$ in the trace norm then put $\phi(y) = \lim \phi(y_n)$. Correctness of this definition follows from (2.10) with ϕ instead of f. So the A-modules $M^\perp$ and N^* coinside and the theorem is proved because the module N^* is isomorphic to H_A^*. •

Proposition 2.5. *Let $N \subset H_B^*$ be a Hilbert submodule over a W^*-algebra B and let $N^\perp = 0$. Then its dual module N^* coinsides with H_B^*.*

Proof. According to supposition for any $z \in H_B^*$ there exists some $x \in N$ such that $\langle z, x \rangle \neq 0$. Therefore the map $z \longmapsto \langle z, \cdot \rangle$ defines the monomorphism $j^* : H_B^* \longrightarrow N^*$ which is dual to the inclusion $j : N \hookrightarrow H_B^*$ after identification of H_B^* and its dual $(H^*)^*$. Their composition

$$j^* \circ j : N \longrightarrow H_B^* \longrightarrow N^*$$

coinsides with the natural inclusion $N \hookrightarrow N^*$. Its dual map

$$(j^* \circ j)^* = j^* \circ j^{**} : N^* = N^{**} \longrightarrow H_B^* \longrightarrow N^*$$

must be an isomorphism, therefore j^* must be epimorphic. •

Proposition 2.6. *Let B be a W^*-algebra and let $R \subset H_B$ be a B-submodule without orthogonal complement, i.e. $R^\perp = 0$ in H_B. Then $R^* = H_B^*$.*

Proof. It is easy to verify that if $R \subset H_B$ then $R^* \subset H_B^*$. As the module R^* is autodual, so by [4] R^* is orthogonally complementary, therefore $H_B = R^* \oplus S$ with some B-module S. Notice that the map $H_B \longrightarrow R^*;\ x \longmapsto \langle x, \cdot \rangle$ is monomorphic by supposition. So we have $S \perp H_B$. But as it is known that $H_B^\perp = 0$ in H_B^*, so $S = 0$. •

3 Compact self-adjoint operators in Hilbert A-modules

By $\mathrm{End}_B^*(M)$ we denote the set of all bounded B-linear operators acting on a Hilbert B-module M over a C^*-algebra B and possessing a bounded adjoint operator.

Proposition 3.1. *If B is a W^*-algebra then $\mathrm{End}_B^*(H_B^*)$ is a W^*-algebra.*

Proof is reduced to verification of the isomorphy between $\mathrm{End}_B^*(H_B^*)$ and the W^*-algebraic tensor product of B by the algebra of bounded operators on the separable Hilbert space.

Recall the definition of the compact operators in a Hilbert B-module M. Put $\theta_{x,y}(z) = x\langle y, z\rangle$ for $x, y, z \in M$. Then $\theta_{x,y} \in \mathrm{End}_B^*(M)$. The set $\mathbf{K}(M)$ of compact operators is the norm-closed linear hull of the set of all operators of the form $\theta_{x,y}$. Denote by $L_n(B)$ the Hilbert B-submodule of the modules H_B or H_B^* generated by the first n elements of the standart basis $e_1, \dots, e_n$.

Proposition 3.2. *Let C^*-algebra B be unital. Then an operator $K \in \mathrm{End}_B^*(H_B)$ is compact if and only if the norm of the restriction of K to the orthogonal complement to L_n tends to zero.*

Proof. Denote by P_n the projection $H_B \longrightarrow L_n(B)^\perp$. Then for any $z \perp L_n(B)$ we have

$$\begin{aligned}
\|\theta_{x,y}(z)\|^2 &= \|\langle \theta_{x,y}(z), \theta_{x,y}(z)\rangle\| = \|\langle y, z\rangle^* \langle x, x\rangle \langle y, z\rangle\| \\
&\le \|x\|^2 \|\langle y, z\rangle\|^2 = \|x\|^2 \|\langle P_n y, z\rangle\|^2 \\
&\le \|x\|^2 \cdot \|P_n y\|^2 \cdot \|z\|^2.
\end{aligned}$$

As $\|P_n y\|$ tends to zero, so does the norm of $\theta_{x,y}$ restricted to $L_n(B)^\perp$. The same is true for linear combinations of such operators and for their norm closure. Suppose now that for some operator K we have $\|K|_{L_n(B)^\perp}\| \to 0$. Then as $\sum_{m=1}^n K e_m \langle e_m, z\rangle = 0$ for any $z \perp L_n(B)$, so if $\|z\| \le 1$ and $z \perp L_n(B)$ then we have

$$\sup_z \|Kz - \sum_{m=1}^n K e_m \langle e_m, z\rangle\| = \sup_z \|Kz\| \longrightarrow 0 \tag{3.1}$$

when $n \longrightarrow \infty$. If $z \in L_(B)$ then $Kz = \sum_{m=1}^{n} Ke_m \langle e_m, z \rangle$. It means that (3.1) holds also if the supremum is taken in the unit ball of the whole H_B, therefore the operator K is the norm topology limit of the operators $K_n = \sum_{m=1}^{n} \theta_{Ke_m, e_m}$.[3] $\bullet$

Remark 3.3. This property of the compact operators was taken as their definition in [13]. Without the supposition that B is unital these two definitions fail to be equivalent. As it was shown in [5] the property of an operator to be compact strongly depends on the choice of a Hilbert structure. Throughout this paper we consider only the standart Hilbert structure on H_B.

Let K be a self-adjoint compact operator acting in H_A. Due to its self-adjointness this operator can be prolonged to an operator K^* in H_A^*.

Lemma 3.4. *If* $\operatorname{Ker} K = 0$ *in* H_A *then* $\operatorname{Ker} K^* = 0$ *in* H_A^*.

Proof obviously follows from the proposition 2.6. One must take the norm closure of $\operatorname{Im} K$ in H_A as R. Then $\operatorname{Ker} K = R^{\perp} = 0$, hence $R^* = H_A^*$ and $\operatorname{Ker} K^* = (R^*)^{\perp} = 0$. $\bullet$

For now on we shall not distinguish the operator K and its prolongation K^* and denote both of them by K.

Now we shall produce an example which shows the necessity of consideration of the dual Hilbert modules if we want to diagonalize compact operators.

Example 3.5. Let $A = L^{\infty}([0;1])$ and let b_k be a monotonous sequence of positive numbers converging to zero. Put

$$a_k = \begin{cases} 1, & t \in (\frac{1}{2^k}; \frac{1}{2^{k-1}}], \\ 0, & \text{for other} \quad t, \end{cases}$$

and put $f_k(t) = b_k \cdot a_k(t)$. Let K be a compact operator which can be written in the form

$$K = \begin{pmatrix} f_1 & f_2 & \cdots & f_n & \cdots \\ f_2 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & & \vdots & \\ f_n & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & & \vdots & \end{pmatrix}$$

in the standart basis of H_A. One can easily diagonalize pointwise this operator. Then the eigenvector corresponding to the maximal eigenvalue can be written as $x = (x_n(t))$ with $x_1(t) = a_1(t) + \frac{\sqrt{2}}{2} \sum_{k>1} a_k(t)$, and $x_n = \frac{\sqrt{2}}{2} a_n(t)$ when $n > 1$. Then $\langle x, x \rangle = \sum_k a_k(t) = 1$. This series converges but not in the norm topology of A, so we have $x \in H_A^* \setminus H_A$.

4 Diagonalization of compact operators in H_A^*

We say that a compact operator K in a Hilbert module M is positive if for any $x \in M$ the operator $\langle Kx, x\rangle \in A$ lies in the positive cone of A. In Hilbert modules as well as in Hilbert spaces positive operators are self-adjoint. A set of elements $\{x_i\} \in H_A^*$ we call a *"basis"* if $\langle x_i, x_i\rangle = \delta_{ij}$ and if the dual A-module for the module generated by this set coinsides with H_A^*, i.e. $(\mathrm{Span}_A\{x_i\})^* = H_A^*$. Notice that a "basis" is neither algebraic nor topological basis. An element $x \in H_A^*$ we call an *"eigenvector"* and an operator $\lambda \in A$ we call an *"eigenvalue"* for K if x generates a projective A-module and $Kx = x\lambda$.

Theorem 4.1. *Let K be a compact positive operator in H_A^* with* $\operatorname{Ker} K = 0$*. Then there exists a "basis" $\{x_i\}$ in H_A^* consisting of "eigenvectors", i.e. $Kx_i = x_i\lambda_i$ for some "eigenvalues" $\lambda_i \in A$.*

Proof. The W^*-algebra $\mathrm{End}_A^*(H_A^*)$ is semifinite and its center is the same as the center Z of A, so this algebra as well as A can be decomposed into a direct integral of factors over the compact Borel space Γ with the finite measure $d\gamma$ such that $L^\infty(\Gamma) = Z$. The operator K then also can be decomposed,

$$K = \int_\Gamma^\oplus K(\gamma)\, d\gamma.$$

If we put $\bar T = T \otimes \mathrm{tr}$ where T is the standart Z-valued finite trace on A and tr is the standart trace in the Hilbert space we obtain a semifinite center-valued trace on the W^*-algebra $\mathrm{End}_A^*(H_A^*)$. At first we show that if we separate the spectrum of K from zero then we find ourselves in the finite trace ideal of $\mathrm{End}_A^*(H_A^*)$. Let χ_E denote as usual the characteristic function of a set $E \subset \mathbf{R}$.

Lemma 4.2. *For every $\varepsilon > 0$ almost everywhere on Γ we have $\bar T(\chi_{(\varepsilon;+\infty)}(K)) < \infty$.*

Proof. Denote the spectral projection $\chi_{(\varepsilon;+\infty)}(K)$ by P. Then the operator inequality

$$K|_{\mathrm{Im}\, P} \geq \varepsilon \tag{4.1}$$

is satisfied on the A-submodule $\mathrm{Im}\, P \subset H_A^*$ by the spectral theorem. Due to compactness of K we can decompose H_A into a direct sum: $H_A = L_n(A) \oplus R$ with such number n that $\|K|_R\| < \varepsilon$. If we pass on to the dual modules then we obtain the estimate

$$\|K|_{R^*}\| < \varepsilon \tag{4.2}$$

where $H_A^* = L_n(A) \oplus R^*$. Denote by Q the projection in H_A^* onto R^*. Then the projection onto $\mathrm{Im}\, P \cap R^*$ will be $P \wedge Q$ and the projection onto $(\operatorname{Ker} P \cap L_n(A))^\perp$ will be $P \vee Q$. By the results of [23] we have

$$\bar T(P \vee Q) = \bar T(P - P \wedge Q). \tag{4.3}$$

As $P \vee Q \leq 1$ where 1 stands for the unity operator in $\mathrm{End}^*_A(H^*_A)$, so we obtain the inequality $\bar{T}(P \vee Q - Q) \leq \bar{T}(1 - Q)$. But $1 - Q$ is the projection onto $L_n(A)$ and its trace is equal to n, so from (4.3) we have $\bar{T}(P - P \wedge Q) \leq n$. Comparing (4.1) with (4.2) we conclude that $\mathrm{Im}\, P \cap R^* = 0$, so $P \wedge Q = 0$ and finally we have $\bar{T}(P) \leq n$. •

We shall need subsequently one simple fact concerning measurable functions.

Lemma 4.3. *Let Γ be a Borel space with a measure and let $\psi : \Gamma \times \mathbf{R} \longrightarrow \mathbf{R}$ be such function that*
(i) for every $\lambda \in \mathbf{R}$ the function $\psi(\gamma; \lambda)$ is measurable on Γ;
(ii) $\psi(\gamma; \lambda)$ is right-continuous and monotonely non-increasing in the second argument for almost all γ.
For any real α put

$$c_\alpha(\gamma) = \inf\{\lambda : \psi(\gamma; \lambda) \leq \alpha\}. \tag{4.4}$$

Then the function $c_\alpha(\gamma)$ is measurable.

Proof. We have to show that for any $\beta \in \mathbf{R}$ the set $V = \{\gamma : c_\alpha(\gamma) \leq \beta\}$ must be measurable. But from the definition of $c_\alpha(\gamma)$ and from (ii) we have $V = \{\gamma : \inf\{\lambda : \psi(\gamma; \lambda) \leq \alpha\} \leq \beta\} = \{\gamma : \psi(\gamma; \beta) \leq \alpha\}$. By (i) we are done. •

Recall that the operator K is decomposable over Γ. Let

$$P_1(\gamma; \lambda) = \chi_{(\lambda; +\infty)}(K(\gamma));$$

$$P_2(\gamma; \lambda) = \chi_{[\lambda; +\infty)}(K(\gamma))$$

be the spectral projections of the operator $K(\gamma)$ corresponding to the sets $(\lambda; +\infty)$ and $[\lambda; +\infty)$ respectively. Put

$$P_1(\lambda) = \chi_{(\lambda; +\infty)}(K); \qquad P_2(\lambda) = \chi_{[\lambda; +\infty)}(K)$$

and $\phi(\gamma; \lambda) = \bar{T}(P_1(\lambda))$. Then this function satisfies the conditions of the lemma 4.3, therefore the function

$$\lambda(\gamma) = \inf\{\lambda : \phi(\gamma; \lambda) \leq 1\} \tag{4.5}$$

is measurable.

Now we want to define two new projections in H^*_A:

$$\begin{aligned} P_1 &= \int_\Gamma^\oplus \chi_{(\lambda(\gamma); +\infty)}(K(\gamma))\, d\gamma = \int_\Gamma^\oplus P_1(\gamma; \lambda(\gamma))\, d\gamma; \\ P_2 &= \int_\Gamma^\oplus \chi_{[\lambda(\gamma); +\infty)}(K(\gamma))\, d\gamma = \int_\Gamma^\oplus P_2(\gamma; \lambda(\gamma))\, d\gamma \end{aligned} \tag{4.6}$$

and we have to check correctness of this definition.

Lemma 4.4. *The operator-valued functions* $P_1(\gamma;\lambda(\gamma))$ *and* $P_2(\gamma;\lambda(\gamma))$ *are measurable.*

Proof. It is understood that the W^*-algebra A is acting on the direct integral of Hilbert spaces $H=\int_\Gamma^\oplus H(\gamma)\,d\gamma$ with the scalar product $(\cdot,\cdot)$. We have to show that the function

$$\gamma \longmapsto \Big(P_1(\gamma;\lambda(\gamma))\,\xi(\gamma),\xi(\gamma)\Big) \tag{4.7}$$

is measurable for all $\xi=\int_\Gamma^\oplus \xi(\gamma)\,d\gamma\in H$. By the theorem XIII.85 of [21] the function

$$\psi(\gamma;\lambda)=\Big(P_1(\gamma;\lambda(\gamma))\,\xi(\gamma),\xi(\gamma)\Big)$$

satisfies the conditions of the lemma 4.3. Measurability of (4.7) follows from measurability of the set $U=\{\gamma:\psi(\gamma;\lambda(\gamma))\le\alpha\}$ for every α. But from the definition of the function $c_\alpha(\gamma)$ (4.4) one can see that

$$U=\{\gamma:\lambda(\gamma)\ge c_\alpha(\gamma)\}=\{\gamma:\lambda(\gamma)-c_\alpha(\gamma)\ge 0\}.$$

This set is measurable because of the measurability of function $\lambda(\gamma)-c_\alpha(\gamma)$. The case of the second projection P_2 can be handled in the same way. •

Corollary 4.5. *The projections* P_1 *and* P_2 *(4.5) are well-defined and* $\bar T(P_1)\le 1;\ \bar T(P_2)\ge 1;\ P_1\le P_2$.

These two projections define the decomposition of H_A^* into three modules:

$$H_A^*=H_-\oplus H_0\oplus H_+ \tag{4.8}$$

where $H_+=\operatorname{Im}P_1;\ H_0=\operatorname{Im}(P_2-P_1);\ H_-=\operatorname{Ker}P_2$. The operator K commutes with these projections because $K(\gamma)$ commutes with the projections $P_1(\gamma;\lambda(\gamma))$ and $P_2(\gamma;\lambda(\gamma))$ for almost all γ, so with respect to the decomposition (4.8) K can be written in the form

$$K=\begin{pmatrix} K_+ & 0 & 0\\ 0 & K_0 & 0\\ 0 & 0 & K_-\end{pmatrix}$$

and K_0 for almost all γ is the operator of multiplication by a scalar $\lambda(\gamma)$, hence every submodule of H_0 is invariant for K. From the corollary 4.5 we can conclude that there exists a projection P such that $P_1\le P\le P_2$ and $\bar T(P)=1$. Then the operator K is diagonal also with respect to the decomposition $H_A^*=\operatorname{Im}P\oplus\operatorname{Ker}P$:

$$K=\begin{pmatrix} K_1 & 0\\ 0 & K'\end{pmatrix}.$$

Notice that the module $\operatorname{Im} P$ is isomorphic to A because the projections onto them in H_A^* have the same trace $\bar{T}$, hence they are equivalent [23]. Let $x_1 \in H_A^*$ be a generator of the module $\operatorname{Im} P$, $\langle x_1, x_1 \rangle = 1$. If it is fixed then the operator $K_1 : \operatorname{Im} P \longrightarrow \operatorname{Im} P$ can be viewed as the operator of multiplication by some $\lambda_1 \in A$; $K_1 x_1 a = x_1 \lambda_1 a$ for $a \in A$, $x_1 a \in \operatorname{Im} P$. By the theorem 2.1 the module $\operatorname{Ker} P$ is isomorphic to H_A^* and the operator K' is obviously compact on $\operatorname{Ker} P$ and the lemma 4.2 holds for it. Moreover we have the operator inequality $K_1 = \lambda_1 \geq K'$.

Further on by induction we can find elements $x_i \in H_A^*$ with $\langle x_i, x_j \rangle = \delta_{ij}$ and operators $\lambda_i \in A$ such that $K x_i = x_i \lambda_i$ and $\lambda_{i+1} \leq \lambda_i$. Denote by N the A-module generated by these elements x_i. Obviously $N \cong H_A$. Notice that the operator $K|_N$ need not to be compact. It remains to show that $N^* = H_A^*$.

Lemma 4.6. *The norm of the operators λ_i tends to zero.*

Proof. Since the sequence $\|\lambda_i\|$ is monotonously non-increasing it converges to some number $b \geq 0$. Suppose that $b \neq 0$. The operators λ_i as well as the other objects involved can be decomposed into direct integrals over Γ. From construction of λ_i we can conclude that there exist such numbers $d_i(\gamma)$ that

$$\lambda_i(\gamma) \geq d_i(\gamma) \geq \lambda_{i+1}(\gamma) \tag{4.9}$$

If we decompose x_i into a direct integral coordinatewise: $x_i = \int_\Gamma^\oplus x_i(\gamma)\, d\gamma$ then for almost all γ $x_i(\gamma)$ are orthonormal in H_A^* and $K(\gamma) x_i(\gamma) = x_i(\gamma) \lambda_i(\gamma)$. Define a function $b(\gamma)$ as the limit of the norms $\|\lambda_i(\gamma)\|$ taken in $A(\gamma)$. We have

$$\|\lambda_i(\gamma)\| = \|\langle K(\gamma) x_i(\gamma), x_i(\gamma) \rangle\| \geq b(\gamma), \tag{4.10}$$

where the inner product is also taken in the $A(\gamma)$-modules $H_{A(\gamma)}^*$. Let now x be an element of N. Then it can be written in the form

$$x(\gamma) = \sum_i x_i(\gamma) a_i(\gamma) \quad \text{with some} \quad a_i = \int_\Gamma^\oplus a_i(\gamma)\, d\gamma \in A.$$

If $\langle x, x \rangle = 1$ then for almost all γ

$$\sum_i a_i^*(\gamma) a_i(\gamma) = 1. \tag{4.11}$$

From (4.9) and (4.10) we can conclude that for all i the operator inequality $\lambda_i(\gamma) \geq b(\gamma)$ holds. Therefore

$$\begin{aligned} \langle K(\gamma) x(\gamma), x(\gamma) \rangle &= \sum_i a_i^*(\gamma) \lambda_i(\gamma) a_i(\gamma) \\ &\geq \sum_i a_i^*(\gamma) a_i(\gamma) b(\gamma) \;=\; b(\gamma) \end{aligned}$$

due to (4.11) and $b(\gamma)$ being a scalar. Further on we obtain that

$$\|\langle Kx, x\rangle\| = \text{ess sup}\,\|\langle K(\gamma)x(\gamma), x(\gamma)\rangle\| \geq \text{ess sup}\, b(\gamma) = b$$

and as by supposition $b > 0$, so

$$\|\langle Kx, x\rangle\| \geq b \tag{4.12}$$

for any $x \in N$ with $\langle x, x\rangle = 1$. Now consider the projection $P_n : N \longrightarrow L_n(A)$. If the spectrum of this operator would be separated from zero then P_n would be an inclusion of the module N into the module L_n, but it is impossible for finite W^*-algebras. Therefore for any $\varepsilon > 0$ we can find $x \in N$ with $\langle x, x\rangle = 1$ such that $\|P_n x\| < \varepsilon$. Put $x' = P_n x$; $x'' = x - x'$. We have $\|x'\| < \varepsilon$; $\|x''\| \leq 1$. Estimate the norm of $\langle Kx, x\rangle$:

$$\begin{aligned}
\|\langle Kx, x\rangle\| &\leq \|\langle Kx', x'\rangle\| + 2\,\|\operatorname{Re}\langle Kx', x''\rangle\| + \|\langle Kx'', x''\rangle\| \\
&\leq \|K\|\,\|x'\|^2 + 2\,\|K\|\,\|x'\|\,\|x''\| + \|\langle Kx'', x''\rangle\| \\
&\leq \|K\|\varepsilon^2 + 2\,\|K\|\varepsilon + \|\langle Kx'', x''\rangle\|.
\end{aligned}$$

As $x'' \perp L_n$, so due to compactness of K we have $\|\langle Kx'', x''\rangle\| < \varepsilon$ for n great enough. Hence $\|\langle Kx, x\rangle\| < \varepsilon'$ where $\varepsilon' = \|K\|\varepsilon^2 + 2\|K\|\varepsilon + \varepsilon$. Choosing ε small enough this estimate contradicts (4.12), so our supposition $b > 0$ is wrong. •

We have proved that the norm of the restriction of K to the orthogonal complement to $x_1, \ldots, x_n$ tends to zero. It means that if $x \in N^\perp$ then $\|Kx\| = 0$. But $\operatorname{Ker} K = 0$, so $N^\perp = 0$ and by the proposition 2.5 we have $N^\perp = H_A^*$. •

The "eigenvalues" λ_i of K are obviously not uniquely determined and the same is true for the "eigenvectors" x_i. If for example we take $x'_i = x_i u_i$ with unitaries $u_i \in A$ then the "eigenvalues" of K will be the operators $\lambda'_i = u_i^* \lambda_i u_i$. The other reason of non-uniqueness is absence of order relation even in commutative W^*-algebras. For example if $A = L^\infty(X)$ and if $\lambda_i = f(x)$; $\lambda_j = g(x)$ are such functions that for some x $f(x) > g(x)$ and for some other x the inverse inequality holds then the functions $\max(f(x), g(x))$ and $\min(f(x), g(x))$ are also "eigenvalues". Nevertheless the next proposition shows that putting the "eigenvalues" in some order provides their uniqueness.

Proposition 4.7. *Let λ_i and x_i be as constructed in the theorem 4.1, and let μ_i be the "eigenvalues" of K corresponding to another "basis" $\{y_i\}$ of H_A^*. If for any unitaries $v_i \in A$ and for all i we have $v_i^* \mu_i v_i \geq v_{i+1}^* \mu_{i+1} v_{i+1}$ then λ_i and μ_i coinside up to unitary equivalence.*

Proof. One can easily check that by supposition we have $\inf \operatorname{Sp} \mu_i(\gamma) \geq \sup \operatorname{Sp} \mu_{i+1}(\gamma)$ in factor $A(\gamma)$ for almost all $\gamma \in \Gamma$. So the projections in H_A^*

onto the modules generated by y_i are spectral projections for K. Denote the projection onto $\mathrm{Span}_A(y_1)$ by Q. Then obviously $P_1 \le Q \le P_2$ where P_1, P_2 are defined by (4.6). We can decompose Q into the sum $Q = P_1 \oplus R$ and the projection P onto $\mathrm{Span}_A(x_1)$ into the sum $P = P_1 \oplus S$ where R and S are also projections. As $\bar{T}(P) = \bar{T}(Q) = 1$, so R and S are equivalent and $\mathrm{Im}\, R \cong \mathrm{Im}\, S$. This module isomorphism commutes with the action of K because the restriction of K onto these modules is scalar and coinsides with $d_1(\gamma)$ for almost all $\gamma \in \Gamma$. So there exists a unitary $u_1 \in A$ realizing this isomorphism between $\mathrm{Im}\, P$ and $\mathrm{Im}\, Q$ such that $\lambda_1 = u_1^* \mu_1 u_1$. Acting by induction we obtain unitary equivalence of the two sets of "eigenvalues". $\bullet$

In the end of this section we must say a few words about diagonalization theorem in the case if we drop out requests about positiveness and absence of kernel for K. If K is any compact operator in H_A or in H_A^* then H_A^* can be decomposed into a direct sum $H_A^* = H_- \oplus \mathrm{Ker}\, K \oplus H_+$ so that the restriction of K onto H_+ (resp. H_-) is positive (resp. negative). We can find sets of "eigenvectors" independently in H_+ and in H_- but we need to drop out the demand for these "eigenvectors" to be units, i.e. the inner squares of such vectors are some projections but not necessarily unities. It is shown in [6] that any compact self-adjoint operator acting in an autodual Hilbert module over a W^*-algebra can be diagonalized, but its "eigenvectors" are not units and its "eigenvalues" are not uniques up to unitary equivalence.

5 Quadratic forms on H_A^* related to self-adjoint operators

Quadratic forms play an important role in the classical operator theory in Hilbert spaces. If B is a C^*-algebra with a faithful finite trace τ and D is a self-adjoint operator acting on a Hilbert B-module M then a quadratic form on M can be defined as $Q(x) = \tau\Big(\langle Dx, x\rangle\Big)$ for $x \in M$. We shall see in this section that this C^*-module quadratic form behaves itself like a usual one. Recall that by $B_1(M)$ we denote the unit ball of M.

Proposition 5.1. *Let D be a positive operator in M with* $\mathrm{Ker}\, D = 0$ *and let the quadratic form $Q(x)$ reach its supremum on $B_1(M)$ at some vector x. Then $\langle x, x\rangle$ is a projection.*

Proof. Denote $\langle x, x\rangle$ by $h \in A$. By definition we have $\|h\| \le 1; h > 0; h^* = h$. Suppose that the spectrum of h contains some number c besides zero and unity. Define a function $f(t)$ on $[0;1] \supset \mathrm{Sp}\, h$ by

$$f(t) = \begin{cases} \frac{1}{\sqrt{\varepsilon}}, & 0 \le t \le \varepsilon, \\ \frac{1}{\sqrt{t}}, & \varepsilon \le t \le 1, \end{cases}$$

where $0 < \varepsilon < c$. Put $a = f(h)$ and $x' = xa$. Then $\langle x', x'\rangle = aha = ha^2$. This operator is equal to the value of the function $t \cdot f^2(t)$ calculated for the operator h. As $t \cdot f^2(t) \le 1$ for $0 \le t \le 1$, so $\|ha^2\| \le 1$ and x' lies in $B_1(M)$. By supposition $\langle Dx, x\rangle$ is a positive operator. Denote it by k^2 with $k \ge 0$. Then

$$\begin{aligned} Q(x') - Q(x) &= \tau\Big(\langle Dxa, xa\rangle - \langle Dx, x\rangle\Big) \;=\; \tau(ak^2a - k^2) \\ &= \tau(a^2k^2 - k^2) \;=\; \tau(ka^2k - k^2) \\ &= \tau(k(a^2-1)k). \end{aligned}$$

By definition $a^2 - 1$ is also positive and we denote it by b^2 with $b \ge 0$. Then

$$\tau(k(a^2-1)k) = \tau(kb^2k) = \tau\Big(b\langle Dx, x\rangle b\Big) = \tau\Big(\langle Dxb, xb\rangle\Big)$$

and thus

$$Q(x') - Q(x) = \tau\Big(\langle Dxb, xb\rangle\Big) \tag{5.1}$$

But $\langle xb, xb\rangle = b\langle x, x\rangle b = bhb$ and the operator bhb corresponds to the function $t(f(t) - 1)$. This function differs from zero when $t = c$, therefore the operator bhb differs from zero, and it means that xb also differs from zero. Notice that the operator $D^{1/2}$ is well-defined and $\operatorname{Ker} D^{1/2} = 0$. Therefore

$$\tau\Big(\langle Dxb, xb\rangle\Big) = \tau\Big(\langle D^{1/2}xb, D^{1/2}xb\rangle\Big) > 0,$$

hence from (5.1) we obtain the inequality $Q(x') - Q(x) > 0$ and it contradicts the supposition that $Q(x)$ is the supremum of the quadratic form Q on $B_1(M)$. So we have proved that $\operatorname{Sp} h$ does not contain any other number except zero and unity, hence h is a projection. •

Proposition 5.2. *Let $x \in B_1(M)$ be a vector at which the quadratic form Q reaches its supremum on B_1 and let $L \subset M$ be a submodule generated by x. Then $M = L \oplus L^\perp$ and L and $L^\perp$ are D-invariant submodules, i.e. $DL \subset L$; $DL^\perp \subset L^\perp$.*

Proof. By the previous proposition $\langle x, x\rangle$ is a projection, hence L is a projective module and by the Dupré-Fillmore theorem [3] $M = L \oplus L^\perp$. Let $y \in L^\perp$; $\|y\| = 1$. Put $x_t = x\cos t + y \sin t$. As $x_0 = x$ is a point of maximum for Q, so $\frac{d}{dt}Q(x_t) = 0$ when $t = 0$. It is easy to see that

$$\frac{d}{dt}Q(x_t)|_{t=0} = \tau\Big(\langle Dx, y\rangle + \langle Dx, y\rangle^*\Big).$$

If $\langle Dx, y\rangle \ne 0$ then put $a = \frac{1}{\|\langle Dx,y\rangle\|}\langle Dx, y\rangle$. Obviously $\|a\| = 1$. Put further $z = ya^*$ and $\bar{x}_t = x\cos t + z\sin t$. Then

$$\begin{aligned} 0 &= \frac{d}{dt}Q(\bar{x}_t)|_{t=0} \;=\; \tau\Big(\langle Dx, z\rangle + \langle Dx, z\rangle^*\Big) \\ &= \tau\Big(\langle Dx, y\rangle a^* + a\langle Dx, y\rangle^*\Big) \;=\; \tau\Big(2aa^* \cdot \|\langle Dx, y\rangle\|\Big) \\ &= 2\,\|\langle Dx, y\rangle\| \cdot \tau(aa^*). \end{aligned}$$

From the faithfulness of τ we obtain $a = 0$, hence Dx is orthogonal to any $y \in L^\perp$, so $Dx \in L$. By self-adjointness of D we have $\langle Dx, y\rangle = 0$ for any $y \in L^\perp$, so $DL^\perp \subset L^\perp$. $\bullet$

Proposition 5.3. *Let $x \in B_1(H_B)$ be a vector at which the quadratic form Q reaches its supremum on $B_1(H_B)$. Then $\langle x, x\rangle = 1$.*

Proof. If $\langle x, x\rangle$ is less than unity then there exists $y \in x^\perp$ such that $\|y\| \leq 1$, $y \neq 0$ and $yq = y$ where $q = 1 - \langle x, x\rangle$ is a projection. Then $\langle x + y, x+y\rangle = \langle x, x\rangle + \langle y, y\rangle \leq 1$, so $x+y \in B_1(H_B)$. But $Q(x+y) = Q(x) + Q(y)$ by the previous proposition and as $y \neq 0$ and $\operatorname{Ker} D = 0$, so $Q(y) > 0$, hence $Q(x+y) > Q(x)$. This contradiction proves the proposition. $\bullet$

We call an operator D in M diagonalizable if it possesses a "basis" consisting of "eigenvectors".

Proposition 5.4. *Let an operator D in M be positive and diagonalizable. If for its "eigenvalues" one has $\operatorname{Sp}\lambda_i \geq \operatorname{Sp}\lambda_{i+1}$ then the supremum of the quadratic form Q on $B_1(M)$ is reached at the first "eigenvector" x_1 and is equal to $\tau(\lambda_1)$.*

Proof. Any $x \in B_1(M)$ can be decomposed: $x = \sum_i x_i a_i$ with $a_i \in B$. Then

$$\begin{aligned} Q(x) &= \tau\Big(\langle Dx, x\rangle\Big) = \tau\Big(\sum_i a_i^* \langle Dx_i, x_i\rangle a_i\Big) \\ &= \tau\Big(\sum_i a_i^* \lambda_i a_i\Big) \leq \tau(a_1^*\lambda_1 a_1) + \sum_{i>1} \tau(a_i^* \lambda_i a_i). \end{aligned}$$

Let $\operatorname{Sp}\lambda_1 \geq d \geq \operatorname{Sp}\lambda_2$. Then

$$Q(x) = \tau(a_1^*\lambda_1 a_1) + \sum_{i>1} \tau(a_i^* d a_i) \leq \tau(a_1^*\lambda_1 a_1) + d\tau(1 - a_1^* a_1)$$

because the inequality $\sum_i a_i^* \leq 1$ follows from $\|x\| \leq 1$. Further on

$$\begin{aligned} Q(x) &\leq \tau(a_1^*\lambda_1 a_1) + d(1 - \tau(a_1^* a_1)) = \tau(a_1^*\lambda_1 a_1 - a_1^* d a_1) + d \\ &= \tau(a_1^*(\lambda_1 - d)a_1) + d = \tau\Big((\lambda_1 - d)^{1/2} a_1 a_1^* (\lambda_1 - d)^{1/2}\Big) + d \\ &\leq \|a_1\|^2 \cdot \tau(\lambda_1 - d) + d \leq \tau(\lambda_1 - d) + d = \tau(\lambda_1). \end{aligned}$$

So $\tau(\lambda_1)$ is the supremum of $Q(x)$ on $B_1(M)$ and it is reached on x_1. $\bullet$

6 Perturbated Schrödinger operator with irrational magnetic flow as an operator acting in a Hilbert module

In this section we consider the perturbated Schrödinger operator with irrational magnetic flow

$$\left(i\frac{\partial}{\partial x}+2\pi\theta y\right)^2-\frac{\partial^2}{\partial y^2}+W(x,y) \tag{6.1}$$

with a double-periodic perturbation $W(x,y)=W(x+1,y)=W(x,y+1)$. This operator has been studied in a number of papers (see [11],[18]). Applying to the operator (6.1) the Fourier transform in the variable x $(x\longrightarrow\xi)$ and the change of variables: $t=-\frac{\xi}{2\pi}+\theta y$; $s=\frac{\xi}{2\pi}$ we obtain the operator

$$D=\Delta+W \tag{6.2}$$

with

$$\Delta=\theta^2\left(\left(\frac{2\pi t}{\theta}\right)^2-\frac{\partial^2}{\partial t^2}\right) \tag{6.3}$$

and

$$W=\sum_{k,l}w_{kl}\,T_t^k\,T_s^{-k}e^{2\pi ilt/\theta}e^{2\pi ils/\theta}$$

where T_t (resp. T_s) denotes the unit translation in variable t (resp. s), $T_t\,\phi(t,s)=\phi(t+1,s)$, and w_{kl} denote the Fourier series coefficients of the function $W(x,y)$. We suppose that the function $W(x,y)$ is such that $\sum_{k,l}|w_{kl}|<\infty$. Let A_θ be the C^*-algebra generated by two non-commuting unitaries U and V such that $UV=e^{2\pi i\theta}VU$ [1],[2] and let $A_\theta^\infty\subset A_\theta$ be its "infinitely smooth" subalgebra of elements of the form $\sum_{k,l}a_{kl}U^kV^l$ where coefficients a_{kl} are of rapid decay. The Schwartz space $S(\mathbf{R})$ of functions of rapid decay on $\mathbf{R}$ can be made [2] a projective right A_θ^∞-module with one generator. We denote this module by M^∞. The action of A_θ^∞ on M^∞ is given by formulas

$$(\phi U)(t)=\phi(t+\theta);\quad (\phi V)(t)=e^{2\pi it}\phi(t)$$

for $\phi(t)\in M^\infty$. The module M^∞ is generated by a projection $p\in A_\theta^\infty$; $M^\infty\cong pA_\theta^\infty$ with $\tau(p)=\theta$ and as $M^\infty\subset A_\theta$ so M^∞ inherits the norm from A_θ. Its closure $M=M^\infty\otimes_{A_\theta^\infty}A_\theta$ in this norm is a Hilbert A_θ-module. Notice that there exists in $S(\mathbf{R})\subset L^2(\mathbf{R})$ the orthonormal basis $\{\phi_i(t)\}$ consisting of the eigenfunctions of the operator Δ (6.3), and the functions from $S(\mathbf{R}^2)=M^\infty\,\hat{\otimes}\,M^\infty$ can be represented as series $\sum_i\phi_i(t)m_i(s)$ with $m_i(s)\in M^\infty$. Define the A_θ-valued inner product on $S(\mathbf{R}^2)$ by formula

$$\Big\langle\sum_i\phi_i(t)m_i(s),\sum_j\phi_j(t)n_j(s)\Big\rangle=\sum_i\langle m_i(s),n_i(s)\rangle$$

where $n_j(s) \in M^\infty$. By $S(\mathbf{R}; M)$ (resp. $L^2(\mathbf{R}; M)$) we denote the Schwartz space of functions (resp. the space of square-integrable functions) with the values in the Banach space M. The inclusion

$$S(\mathbf{R} \times \mathbf{R}) \hookrightarrow S(\mathbf{R}; M) \hookrightarrow L^2(\mathbf{R}; M) \cong N$$

allows us to consider $S(\mathbf{R}^2)$ as a dense subspace in the Hilbert module

$$N = \{(m_i) : \sum_i \langle m_i, m_i \rangle \text{ converges in } A_\theta\}$$

(this module is often denoted by $l_2(M)$). One can see that the module M is full, i.e. $\langle M, M \rangle = A_\theta$ because the C^*-algebra A_θ is simple and $\langle M, M \rangle$ must be its ideal. By the results of [3] one has $N \cong H_{A_\theta}$.

Theorem 6.1. *The operator D (6.2) is a self-adjoint unbounded operator in N with a dense domain.*

Proof consists of the five following steps.

1. Let $N_1 \subset N$ be a subspace of sequences (m_i) such that the series $\sum_i i^2 \langle m_i, m_i \rangle$ converges in norm to an element of A_θ. If $\xi \in N_1$; $\xi = \sum_i \phi_i(t) m_i$ then $\Delta\xi = \sum_i (2i-1)\theta\phi_i(t) m_i$ and the series $\sum_i (2i-1)^2\theta^2 \langle m_i, m_i \rangle$ converges in A_θ, therefore Δ is an unbounded operator in N with the dense domain N_1.

2. Here we show that the action of the operator $C_{kl} = T_s^{-k} e^{2\pi i l s/\theta}$ can be prolonged from M^∞ to M. Since this operator commutes with the action of the algebra A_θ^∞ on the A_θ^∞-module M^∞ we have $C_{kl} \in \mathrm{End}_{A_\theta^\infty} M^\infty$. The image of the generator p of M^∞ can be written in the form $C_{kl}(p) = p a_{kl} \in M^\infty$ for some $a_{kl} \in A_\theta^\infty$ and we have

$$C_{kl}(p) = C_{kl}(p^2) = C_{kl}(p)p = p a_{kl} p.$$

Obviously the map $m \longmapsto p a_{kl} p m = C_{kl}(m)$ can be continuously prolonged from M^∞ to M. Besides that since C_{kl} is a unitary operator, we have $\|C_{kl}\| = 1$ and $\|a_{kl}\| = 1$.

3. Consider now the operator $B_{kl} = T_t^k e^{2\pi i l t/\theta} \cdot C_{kl}$. It is obviously continuous in $S(\mathbf{R}^2)$. Let α_{ij} be matrix coefficients of decomposition with respect to the basis $\{\phi_j\}$ for the operator $T_t^k e^{2\pi i l t/\theta}$:

$$T_t^k e^{2\pi i l t/\theta} \phi_i(t) = \sum_j \alpha_{ij} \phi_j(t).$$

As this operator is unitary, so $\sum_j \bar{\alpha}_{ij} \alpha_{nj} = \delta_{in}$. Let $\xi = \sum_i \phi_i(t) m_i \in N$. Then $B_{kl}(\xi) = \sum_{i,j} \alpha_{ij} \phi_j(t) C_{kl}(m_i)$. Estimate its norm:

$$\langle B_{kl}(\xi), B_{kl}(\xi) \rangle \;=\; \sum_{i,j} \Big\langle \sum_i \alpha_{ij} C_{kl}(m_i), \sum_n \alpha_{nj} C_{kl}(m_n) \Big\rangle$$

$$
\begin{aligned}
&= \sum_{i,n,j} \bar{\alpha}_{ij}\alpha_{nj}\langle C_{kl}(m_i), C_{kl}(m_n)\rangle \\
&= \sum_{i,n}\Big(\sum_j \bar{\alpha}_{ij}\alpha_{nj}\Big)\langle C_{kl}(m_i), C_{kl}(m_n)\rangle \\
&= \sum_{i,n} \delta_{in}\langle C_{kl}(m_i), C_{kl}(m_n)\rangle \\
&= \sum_i \langle C_{kl}(m_i), C_{kl}(m_i)\rangle \\
&= \sum_i (a_{kl}m_i)^* a_{kl}m_i = \sum_i m_i^* a_{kl}^* a_{kl} m_i \\
&\le \|a_{kl}\|^2 \sum_i m_i^* m_i = \sum_i m_i^* m_i = \langle \xi, \xi\rangle .
\end{aligned}
$$

Hence $\|B_{kl}\| \le 1$ and it is a continuous operator in N.

4. We have

$$
\|W\| \le \sum_{k,l} \|w_{kl}B_{kl}\| \le \sum_{k,l} |w_{kl}| \cdot \|B_{kl}\| \le \sum_{k,l} |w_{kl}|.
$$

By our supposition the last sum is finite, hence W is continuous in N.

5. It remains to show that D commutes with the action of the C^*-algebra A_θ on N. It is obvious for the operators Δ and B_{kl}. As the series $W = \sum_{k,l} w_{kl}B_{kl}$ converges, so W also commutes with the action of A_θ. D is self-adjoint if the function $W(x, y)$ is real-valued. •

Let now A be a type II_1 factor containing A_θ as a weakly dense subalgebra (cf. [1]). This inclusion induces the inclusion of H_{A_θ} into H_A and operators acting in H_{A_θ} can be prolonged to operators acting in H_A. Notice that if $\|W\| < c$ then the operator $D+c$ is invertible and its inverse $(\Delta+W+c)^{-1} = (1 + \Delta^{-1}(W + c))^{-1}\Delta^{-1}$ is compact because the operator Δ^{-1} is compact. So by the theorem 4.1 it is diagonalizable in H_A^*, hence the same is true for the operator D. Slightly changing the proof of that theorem (namely taking θ instead of 1 in (4.5)) we can obtain the set of "eigenvectors" $\{x_i\}$ for D with $\langle x_i, x_i\rangle = p$. In that case the corresponding "eigenvalues" λ_i can be viewed as elements from $\mathrm{End}^*_A(\mathcal{N})$ where $\mathcal{N} = pA = N \otimes_{A_\theta} A$.

Problem 6.2. Can the "eigenvalues" λ_i be taken from the lesser algebra $\mathrm{End}^*_{A_\theta}(M)$ instead of $\mathrm{End}^*_A(\mathcal{N})$? Do these "eigenvalues" possess properties resembling analyticity as they do in the commutative case when θ is integer [18],[21] ?

If $\|W\| < \theta$ then the spectrum of D lies in $\cup_i(2\theta(i-1); 2\theta i)$, therefore the spectral projections $P_i = P_{(2\theta(i-1);\, 2\theta i)}(D)$ lie in $\mathrm{End}^*_{A_\theta}(N)$, hence the "eigenvalues" λ_i of D lie in $\mathrm{End}^*_{A_\theta}(M)$. It was shown in [14] by the methods of

perturbation theory that if the norm of W is small enough then the images of P_i contain "eigenvectors" which form a basis of N, hence the operator D is diagonalizable inside the module N.

Added in proof: The results of [22] allow us to give positive answer to the first question of the problem 6.2.

References

[1] Brenken B.: Representations and automorphisms of the irrational rotation algebra. *Pacif. J. Math.* **111** (1984), 257 – 282.

[2] Connes A.: C^*-algèbres et géométrie différentielle. *C.R. Acad. Sci.,* Paris, **290** (1980), 599 – 604.

[3] Dupré M.J., Fillmore P.A.: Triviality theorems for Hilbert modules. In: Topics in modern operator theory, 5-th Internat. conf. on operator theory. Timisoara and Herculane (Romania), 1980. Basel – Boston – Stuttgart: Birkhäuser Verlag, 1981, 71 – 79.

[4] Frank M.: Self-duality and self-reflexivity of Hilbert C^*-modules. *Zeitschr. Anal. Anw.* **9** (1990), 165 – 176.

[5] Frank M.: Geometrical aspects of Hilbert C^*-modules. Københavns Universitet preprint series 22/1993.

[6] Frank M., Manuilov V.M.: Diagonalizing "compact" operators on Hilbert W^*-modules. Leipzig University, NTZ, Preprint N 10, 1994.

[7] Grove K., Pedersen G.K.: Diagonalizing matrices over $C(X)$. *J. Funct. Anal.* **59** (1984), 64 – 89.

[8] Kadison R.V.: Diagonalizing matrices over operator algebras. *Bull. Amer. Math. Soc.* **8** (1983), 84 – 86.

[9] Kadison R.V.: Diagonalizing matrices. *Amer. J. Math.* **106** (1984), 1451 – 1468.

[10] Kasparov G.G.: Hilbert modules: theorems of Stinespring and Voiculescu. *J. Operator Theory* **4** (1980), 133 – 150.

[11] Lyskova A.S.: On Schrödinger operator in magnetic field. *Usp. Mat. Nauk* **36** (1981), 189 – 190 (in Russian).

[12] Mishchenko A.S.: Banach algebras, pseudo-differential operators and their application to K-theory. *Usp. Mat. Nauk* **34** (1979), 67 – 79 (in Russian).

[13] Mishchenko A.S., Fomenko A.T.: The index of elliptic operators over C^*-algebras. *Izv. Akad. Nauk SSSR. Ser. Mat.* **43** (1979), 831 – 859 (in Russian).

[14] Manuilov V.M.: On eigenvalues of perturbated Schrödinger operator in magnetic field with irrational magnetic flow. *Func. Anal. i Pril.* **28** (1994), 57 – 60 (in Russian).

[15] Manuilov V.M.: Diagonalization of compact operators in Hilbert modules over W^*-algebras of finite type. *Usp. Mat. Nauk* **49** (1994), 159 - 160 (in Russian).

[16] Murphy Q.J.: Diagonality in C^*-algebras. *Math. Zeitschr.* **199** (1990), 279 – 284.

[17] Murray F.J., von Neumann J.: On rings of operators. *Ann. Math.* **37** (1936), 116 – 229.

[18] Novikov S.P.: Two-dimensional Schrödinger operators in periodic fields. In: Modern Problems of Math.,v. 23. Moscow: VINITI, 1983, 3 – 32 (in Russian).

[19] Paschke W.L.: Inner product modules over B^*-algebras. *Trans. Amer. Math. Soc.* **182** (1973), 443 – 468.

[20] Paschke W.L.: The double B-dual of an inner product module over a C^*-algebra. *Canad. J. Math.* **26** (1974), 1272 – 1280.

[21] Reed M., Simon B.: Methods of moderm mathematical physics. 4: Analysis of operators. New-York - San Francisco - London: Academic Press, 1978.

[22] Sunder V.S., Thomsen K.: Unitary orbits of selfadjoints in some C^*-algebras. *Houston J. Math.* **18** (1992), 127 – 137.

[23] Takesaki M.: Theory of operator algebras,1. New-York – Heidelberg – Berlin: Springer Verlag, 1979.

[24] Zhang S.: Diagonalizing projections. *Pacif. J. Math.* **145**(1990), 181 – 200.

V.M.Manuilov
Moscow State Public University
Russia - 129278 Moscow
P.Korchagina str. 22
E-mail:manuilov@math.math.msu.su